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Irregular multiresolution analysis and associated wavelet

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Abstract We introduce two generalizations, the first of which generalizes the concept of multiresolution analysis. We define the irregular generalized multiresolution analysis (IGMRA). This structure is defined taking translations on sets that are not necessarily regular lattices, for which certain density requirements are required, and without using dilations, also allows each subspace of IGMRA to be generated by outer frames of translations of different functions. The second generalization concerns the concept of association of wavelets to these new structures. We take frames of translations of a countable set of functions, which we called *generalized wavelets*, and define the concept of association of these generalized wavelets to those previously defined IGMRA. In the next stage, we prove two existence theorems. In the first theorem, we prove existence of IGMRA, and in the second existence of generalized wavelets associated with it. In the latter, we show that we are able to associate frames of translations with optimal localization properties, to IGMRA. In the last section of this paper, concrete examples of these structures are presented for $L^2(\mathbb{R})$ and for $L^2(\mathbb{R}^2)$.

Mathematics Subject Classification 42C40 · 42C30

الملخص

نقدم تعميمين، يعمِّمُ أولهما مفهوم التحليل متعدد التفصيل. نعرِّف التحليل متعدد التفصيل المُعَمَّم غير المنتظم (IGMRA). يُعَرَّف هذا البناء بأخذ انسحابات على مجموعة ليست بالضرورة شبكاتٍ منتظمة ويتم افتراض متطلبات كثافة معينة لها بدون استخدام أي تمددات، ويسمح لكل فضاءٍ جزئي من IGMRA أن يكون مُوَلَّداً بواسطة إطارات خارجية لانسحابات عدة دوال. يخص التعميم الثاني مفهوم ربط مُوَيْجاتٍ لتلك البناءات الجديدة. نـأخذ إطاراتٍ لانسحابات لمجموعة قابلة للعد من الدوال، والتي أسميناها مُوَيْجاتٍ مُعَمَّمَة، ونعرف مفهوم ربط تلك المُوَيْجاتِ المُعَمَّمَة بِـ IGMRA المعرَّفة مسبقاً. في المرحلة التالية، نثبت مبرهنتي وجود. في المبرهنة الأولى، نثبت وجود IGMRA، ونثبت في الثانية وجود مُوَيْجاتٍ مُعَمَّمَة مرتبطة بها، وأن باستطاعتنا ربط إطاراتٍ من انسحاباتٍ مع خصائص مَوْضعة مُثلى لِـ IGMRA. في الفصل الأخير من هذه الورقة، يتم تقديم أمثلة محددة لتلك البناءات لِـ $L^2(\mathbb{R})$ وَ $L^2(\mathbb{R}^2)$.

1 Introduction

From the classic concept of multiresolution analysis (MRA), introduced and further developed by Meyer [27,28], and Mallat [24,25], which provides a systematic way to construct orthonormal wavelet bases of $L^2(\mathbb{R})$, research in this area has been extended in various ways. These concepts are generalized to $L^2(\mathbb{R}^d)$ [14],

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based on a lemma of Gröechenig [17], to lattices different from $\mathbb{Z}^d$ [31], allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis [34], by changing the original matrix of dilation $2I_d$, by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A\mathbb{Z}^d \subset \mathbb{Z}^d$ [23] (to ensure the existence of orthonormal wavelet basis or Riesz associated with these structures). Then, the original definition of MRA was extended, admitting a finite number of scaling functions (the wavelets associated with these MRA, are called multiwavelets). There is an extensive literature on this subject, for example, [2,9,18,20]. The necessary and sufficient conditions for the existence of wavelets basis associated with these MRA were presented in [8]. Moreover, Baggett et al. [3] extended the definition to Generalized analysis of multiresolution considering an Abelian group of unitary operators Γ over H called *group of translations*, with the property that the subspace V_0 is invariant under the action of Γ. At a later stage, Benedetto and Treiber [6] introduced the MRA frame in $L^2(\mathbb{R}^d)$, for which, the MRA subspaces are generated by wavelet frames (i.e., frames generated by translations and dilations of a single function), and under certain conditions, may be associated with this new structure, frames of all space.

The previous concepts are developed on regular lattices, i.e, $B\mathbb{Z}^d$, being B an invertible matrix. The MRA subspaces are generated by expansion/contraction of an initial subspace, by the powers of an expansive dilation matrix A (A is said to be expansive if $|\lambda| > 1$ for every eigenvalue λ of A). For these MRA, the atomic decomposition of space $L^2(\mathbb{R}^d)$ (the associated wavelets) is usually obtained in terms of frames generated by a family of functions dilated by the powers of the expansive matrix dilation A and shifted on regular grids. These functions generate the initial subspace.

Since long time ago, the irregular sampling problem, has inspired many researchers of mathematics [16,21,22,26,29,32,33]. This theory also owes its increasing development to the interest in solving problems of signal and image processing of different sciences, such as engineering, earth sciences, biology and medical sciences, among others. Wavelet and Gabor frames on irregular grids have been studied by many researchers, for example, in [11–13,30], and more recently in [1,5,10]. In the latter article, construction of nonstationary Gabor frames and direct applications are shown. But in all cases of wavelets on irregular grids, they were not associated with the structure of the MRA.

In this paper, we introduce a new generalization of the traditional MRA structure, admitting to perform translations on *separate sets* of points that require certain density requirements (i.e., discrete sets of points of $\mathbb{R}^d$ are not necessarily regular lattices), and regardless of dilations. Moreover, we also allow the association of a more general structure than traditional wavelets to these MRA, the *generalized wavelets*.

The ideas that gave origin to these new structures emerged from the reading of the paper entitled *Wavelet on irregular grids with arbitrary dilation matrices and frame atoms for* $L^2(\mathbb{R}^d)$ [1], which created the need to find a structure that generalizes the MRA, to which those wavelets exposed in this work are associated. All the concepts in this paper are the results of the doctoral thesis [19]. The theory developed here gives conceptually new mathematical tools, which may promote a significant advancement in signal and image processing.

2 Notation

If $A \in GL_d(\mathbb{R})$, $y \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, we define the unitary operators:

1. $(D_A f)(x) := |\det A|^{1/2} f(Ax)$,
2. $(T_y f)(x) := f(x - y)$
3. $M_y f(x) := e^{-i2\pi\langle y,x\rangle} f(x)$

Furthermore,

- If $f \in L^2(\mathbb{R}^d)$, $\widehat{f}$ and $\check{f}$ will be the Fourier transform and the inverse Fourier transform, respectively. In addition, $e_y(x) := e^{-2\pi i\langle y,x\rangle}$.
- Let $Q \subseteq \mathbb{R}^d$ be a measurable set, then χ_Q is the characteristic function of Q, $\mu(Q)$ to denote the Lebesgue measure of Q and ∂Q will be the boundary of Q.
- Throughout the paper, J and K will denote countable index sets.

3 Preliminary notions

1. **Frames in Hilbert spaces**
Let H be a Hilbert space, and I a countable set of indexes. $(f_i)_{i\in I} \subset H$ is a frame for H, if:
$\exists\ A\ ,\ B:\ 0 < A \leq B < \infty$ so that $\forall f \in H$

$$A\|f\|^2 \leq \sum_{j\in I} |< f, f_i >|^2 \leq B\|f\|^2$$

A and B are the lower and upper bounds of the frame $\{f_i\}_{i\in I}$, and $\{< f, f_i >\}_{i\in I}$ are the coefficients of f in the frame $(f_i)_{i\in I}$.

2. **Outer frame**
$(f_j)_{j\in\mathbb{Z}} \subset H$ (H Hilbert space) is a **outer frame** of the closed subspace F of H, if $(P_F(f_j))_{j\in\mathbb{Z}}$ is a frame of F (P_F is the orthogonal projection on F). This is equivalent to the existence of constants $0 < m \leq M < \infty$, so that:

$$m\|f\|^2 \leq \sum_{j\in\mathbb{Z}} |< f, f_j >|^2 \leq M\|f\|^2 \qquad \forall f \in F$$

The three following definitions were introduced by Aldroubi et al. [1]

Definition 3.1 A *separate set of points* of $\mathbb{R}^d$ is a subset of points $X = \{x_k\}_{k\in\mathbb{Z}}$ so that $\inf_{t\neq s}|x_t - x_s| > 0$.

Definition 3.2 Let $X = \{x_k\}_{k\in Z} \subset \mathbb{R}^d$ be a separate set of points. The *gap* of X is defined by

$$\rho = \rho(X) := \inf\{r > 0\ :\ \cup_{k\in Z} B_r(x_k) = \mathbb{R}^d\}$$

where $B_r(x_k)$ is the ball of the center x_k and radius r.

Definition 3.3 Let Q be a non-empty and measurable subset of $\mathbb{R}^d$. We define the closed subspaces of $L^2(\mathbb{R}^d)$:

$$\mathcal{K}_Q := \{f \in L^2(\mathbb{R}^d) : \text{supp } f \subseteq \overline{Q}\ a.e\}, \quad \text{and}$$

$$\mathcal{B}_Q := \{f \in L^2(\mathbb{R}^d) : \text{supp } \widehat{f} \subseteq \overline{Q}\ a.e\}$$

In dimensions greater than one, Beurling proved [7] the following result:

Theorem 3.4 (Beurling)*: Let $X \subset \mathbb{R}^d$ be a separate set of points, ρ the gap of X, and $\Omega = B_r(0)$. If $r\rho < \frac{1}{4}$, then $\{e_{x_k}\chi_\Omega\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{K}_\Omega$.*

It is easy to observe that $\mathcal{K}_Q = \{\widehat{f} : f \in \mathcal{B}_Q\}$.

The proofs of the following lemmas are directly deduced from the properties of unitary operators and usual results of the theory of frames. These demonstrations can be found in [19].

Lemma 3.5 *Let Q be a measurable subset of $\mathbb{R}^d$. $\{f_j\}_{j\in\mathbb{Z}}$ is a frame for $\mathcal{K}_Q$, with bounds m and M if and only if:*

(i) *$\{D_{A^{-1}} f_j\}_{j\in\mathbb{Z}}$ is a frame for $\mathcal{K}_{AQ}$ with bounds m and M, for all $A \in GL_d(\mathbb{R})$.*
(ii) *$\forall$ and $\in \mathbb{R}^d$ the set $\{T_y f_j\}_{j\in\mathbb{Z}}$ is a frame for $\mathcal{K}_{y+Q}$ with bounds m and M.*
(iii) *$(\check{f}_j)_{j\in\mathbb{Z}}$ is a frame for $\mathcal{B}_Q$, also with bounds m and M.*

Lemma 3.6 *Let V be a measurable and non-empty subset of Q and ∂V the border of V. If $\{f_j\}_{j\in\mathbb{Z}}$ is a frame of $\mathcal{K}_Q$, then:*

(a) *The set of functions $\{f_j\}_{j\in Z}$ is a outer frame of $\mathcal{K}_V$.*
(b) *The set of functions $\{\check{f}_j\}_{j\in Z}$ is a outer frame of $\mathcal{B}_V$.*
(c) *If $\mu(\partial V) = 0$, then $\{f_{j_{|V}}\}_{j\in Z}$ is a frame for $\mathcal{K}_V$ with the same bounds as the frame $\{f_j\}_{j\in Z}$ of $\mathcal{K}_Q$. ($f_{j_{|V}}$ is the restriction to f_j to the set V).*

Lemma 3.7 *For all measurable subset Q of $\mathbb{R}^d$, we have:*
$\{f_j\}_{j\in Z}$ is a outer frame of $\mathcal{K}_Q$ if and only if $\{\check{f}_j\}_{j\in Z}$ is a outer frame of $\mathcal{B}_Q$.

We will state and prove an extremely important result by means of the following demonstrations.

Proposition 3.8 *Let $0 \neq h \in L^2(\mathbb{R}^d)$ and S be a set of positive measure such that $\operatorname{supp} h = \overline{S}$. Let X be a separate set of points in $\mathbb{R}^d$ so that $\{e_{x_k}\chi_S\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{K}_S$. Then:*

(a) $\overline{span}\{he_{x_k}\}_{k\in\mathbb{Z}} = \mathcal{K}_S$.
(b) *If $|h(\xi)| < R < \infty\ \forall\xi \in S$ and $h(\xi) = 0$ in $\mathbb{R}^d\backslash S$, the set $\{he_{x_k}\}_{k\in\mathbb{Z}}$ is a Bessel sequence for the Hilbert space $\mathcal{K}_S$.*
(c) *If $0 < r < |h(\xi)| < R < \infty\ \forall\xi \in S$ and $h(\xi) = 0$ in $\mathbb{R}^d\backslash S$, the set $\{he_{x_k}\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{K}_S$.*
(d) *If h is continuous, the set $\{he_{x_k}\}_{k\in\mathbb{Z}}$ cannot be a frame for $\mathcal{K}_S$.*

For the proof of this proposition, we use the following definitions and results of [4].

Definition 3.9 For two set $F \subseteq E \subseteq \mathbb{R}^d$, let

$$\widetilde{L^2(F)}^{G} = \{f \in L^2(G) : f(x) = 0\ a.e.\ x \in G\backslash F\}.$$

If $G = \mathbb{R}^d$, we write $L^2(F)$ in place of $\widetilde{L^2(F)}^{G}$.

Definition 3.10 Let $E \subseteq \mathbb{R}^d$ and let

$$P(E) = \{f \in L^2(\mathbb{R}^d) :\ \operatorname{supp}\widehat{f} \subseteq E\}.$$

Proposition 3.11 (Proposition 3.2 of [4]) *Let φ be a measurable function defined in $\mathbb{R}^d$ and $\{\psi_k\}_{k\in K}$ be a frame of $L^2(E)$. Then,*

1. *$\{\varphi\psi_k\}_{k\in K}$ is a frame of $L^2(E)$ if and only if there exist constants A and B such that $0 < A \le B < \infty$ and $A \le |\varphi(t)| \le B$ a.e. $t \in E$.*
2. *If $\varphi \in L^2(\mathbb{R}^d)$ such that $\{\varphi\psi_k\}$ is in $L^2(E)$ and $|\{t \in E : \varphi(t) = 0\}| = 0$, then $\{\varphi\psi_k\}$ is complete in $L^2(E)$.*

Proposition 3.12 (Proposition 3.5 of [4]) *Let φ be a measurable function defined in $\mathbb{R}^d$. Let $\{\psi_k\}_{k\in K}$ be a frame of $L^2(E)$. Then, $\{\varphi\psi_k\}_{k\in K}$ is a Bessel sequence of $L^2(E)$ if and only if there exists a constant $B > 0$ such that $|\varphi(t)| \le B$ a.e. $t \in E$.*

Proposition 3.13 (Proposition 3.8 of [4]) *Let $\{\psi_k\}_{k\in K}$ be a frame of $L^2(E)$ and $\varphi \in L^2(\mathbb{R}^d)$ such that $\{\varphi\psi_k\}$ is in $L^2(E)$. Let $F := \operatorname{supp}(\varphi) \bigcap E$. Then,*

1. $\overline{span}\{\varphi\psi_k\} = \widetilde{L^2(F)}^{E}$.
2. *$\{\varphi\psi_k\}_{k\in K}$ is a frame sequence of $L^2(E)$ if and only if there exist constants A and B such that $0 < A \le B < \infty$ and $A \le |\varphi(t)| \le B$ a.e. $t \in F$.*
3. *Let φ be compactly supported. Then, $\{\varphi\psi_k\}_{k\in K}$ is a frame sequence of $L^2(\mathbb{R}^d)$ if and only if there exist constants A and B such that $0 < A \le B < +\infty$ and $A \le |\varphi(t)| \le B$ a.e. $t \in F$.*

Corollary 3.14 (From Theorem 4.1 of [4]) *Let $h \in P_E$ such that $\widehat{h}$ is continuous. Then, there does not exist $\Lambda = \{\lambda_k\}_{k\in K} \subseteq \mathbb{R}^d$ such that $\{h(-\lambda_k)\}_{k\in K}$ is a frame of P_E.*

Proof of Proposition 3.8 We observe that $L^2(\overline{S}) = \mathcal{K}_S$, where $\mathcal{K}_S$ is defined in Definition 3.3. We take $\overline{S} = E$, $h = \varphi$, $\mathbb{Z} = K$ and $\psi_k = e_{x_k}\chi_S\ \forall k$, in the propositions and previous corollary.

(a) Because $\operatorname{supp} h = \overline{S}$ then of the Proposition 3.13 (1.) we deduce that $\overline{span}\{he_{x_k}\}_{k\in\mathbb{Z}} = \mathcal{K}_S$.
(b) We apply the Proposition 3.12 in this demonstration.
(c) We apply the Proposition 3.11 (1). in this demonstration.
(d) We note that $P_S = \mathcal{B}_S$, where $\mathcal{B}_S$ is defined in Definition 3.3. We suppose that $\{he_{x_k}\}_{k\in\mathbb{Z}}$ is a frame of $\mathcal{K}_S$, being h a continuous function, then from Lemma 3.5 (iii) the set $\{\check{h}(-x_k)\}_{k\in\mathbb{Z}}$ is a frame of P_S. This contradicts the Corollary 3.14. □

In the next section, we generalize the definition of MRA, to irregular generalized multiresolution analysis (IGMRA), admitting separate sets of points, not necessarily regular lattices for the translations. The subspaces of the MRA are not related by dilations anymore, but it is allowed that each subspace be generated by outer frames of translations. We then proved the existence of these new structures, and associated them wavelet frames with good localization.

4 Irregular generalized multiresolution analysis

Definition 4.1 Let $X = \{X^j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^d$, $X^j = \{x_k^j\}_{k\in\mathbb{Z}}$ a separate set for each j, and $\mathcal{V} = \{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ a set of closed subspaces of $L^2(\mathbb{R}^d)$. We can say that the pair $\{\mathcal{V}, X\}$ is a **Irregular Generalized Multiresolution Analysis**[1], if

(a) $...\mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset ...$
(b) $\overline{\cup_{j\in\mathbb{Z}}\mathcal{V}_j} = L^2(\mathbb{R}^d)$
(c) $\cap\mathcal{V}_j = 0$
(d) there is a family of functions $\{\varphi_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R}^d)$, so that for each j, the set $\{\varphi_j(x - x_k^j)\}_{k\in\mathbb{Z}}$ is a outer frame of $\mathcal{V}_j$.

We call $\{\varphi_j\}_{j\in\mathbb{Z}}$ *the generating family* of IGMRA.

Remark 4.2 We observe that:

1. Let $\varphi_j = D_{A^j}\varphi$ be, with $\varphi \in L^2(\mathbb{R}^d)$ scaling function of a MRA, determined by an expansive matrix A, such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, and $\forall j \in \mathbb{Z}$ are the lattices $X_j := A^{-j}\mathbb{Z}^d$ $(x_k^j = A^{-j}k\ \forall j,$ and $\forall k \in \mathbb{Z}^d)$; because any frame of a subspace, is also an outer frame for the same subspace, it follows that the classical MRA is a particular case of IGMRA by Definition 4.1.
2. In general, the subspace $\mathcal{V}_j$ cannot be determined from the function φ_j, because this function does not necessarily belong to $\mathcal{V}_j$.
3. In the multiresolution analysis with dilation factor from a matrix expansive A, we have

$$\mathcal{V}_j = D_{A^j}\mathcal{V}_0\ \forall j$$

   In the IGMRA, this property becomes meaningless.

Definition 4.3 Generalized Wavelet of $L^2(\mathbb{R}^d)$
A sequence of pairs $\{\psi_j, X^j\}_{j\in\mathbb{Z}}$ is called a generalized wavelet of $L^2(\mathbb{R}^d)$ if

$$\{\psi_j(. - x_k^j)\}_{j\in\mathbb{Z},k\in\mathbb{Z}} \tag{1}$$

is a frame of $L^2(\mathbb{R}^d)$, generated by the translations of a countable set of functions $\{\psi_j\}_{j\in\mathbb{Z}}$ on a separate set of points $X^j = \{x_k^j\}_{k\in\mathbb{Z}}$.

Remark 4.4 If $\{D_{A^j}T_{x_k^j}\psi\}_{j\in\mathbb{Z};\ k\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R}^d)$, in particular is a generalized wavelet, it suffices to consider $\psi_j = D_{A^j}\psi$ and $X^j = \{A^{-j}x_k^j\}_{k\in\mathbb{Z}}$.

Here, the concept of *association* of wavelets to an IGMRA is extended.

Definition 4.5 Let $\{\mathcal{V}, X\}$ be an IGMRA, with $X = \{X^j\}_{j\in\mathbb{Z}}$, and $\{\psi_j, X^j\}_{j\in\mathbb{Z}}$ a generalized wavelet of $L^2(\mathbb{R}^d)$. We can say that the generalized wavelet is associated with IGMRA $\{\mathcal{V}, X\}$, if for each $j \in \mathbb{Z}$ the frame of translations $\{T_{x_k^j}\psi_j\}_{k\in\mathbb{Z}}$ is a frame (outer frame) of the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$.

What follows below, shows the existence of IGMRA for spaces $L^2(\mathbb{R}^d)$, and of the generalized wavelets associated with it.

4.1 Existence of IGMRA and associated wavelets

Theorem 4.6 *Let $\{U_j\}_{j\in\mathbb{Z}}$ be a collection of open and bounded subsets of $\mathbb{R}^d$ that verify:*

1. $...U_{-1} \subset U_0 \subset U_1 \subset ...$
2. $\mu(\bigcap_j U_j) = 0$
3. $\mu(\partial U_j) = 0\ \forall j \in \mathbb{Z}$

[1] Multiresolution Analysis is *Irregular* in itself, in the event that the separate set is not the regular lattice.

4. $\bigcup_{j\in\mathbb{Z}} U_j = \mathbb{R}^d$
5. *Let* $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ *be a sequence of positive numbers.*

(a) *Let* $U_{\varepsilon_j} := \{x \in \mathbb{R}^d : d(x, U_j) < \varepsilon_j\}$*; and*
(b) $X^j = \{x_k^j\}_{k\in\mathbb{Z}} \subset \mathbb{R}^d$ *be separate sets of points such that* $\{e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{K}_{U_{\varepsilon_j}}$*, with bounds* m_j *and* M_j*. (This is possible because the sets* U_{ε_j} *are open and bounded* [2]*).*
(c) *Let* $\{\varphi_j\}_{j\in\mathbb{Z}} \subset L^2(\mathbb{R}^d)$ *so that:*
 – $\widehat{\varphi_j} \in \mathcal{K}_{U_{\varepsilon_j}}$
 – $0 < c_j \le |\widehat{\varphi_j}(\omega)| \le C_j < \infty\ \forall\omega \in U_j$.

Then, $\{\mathcal{V}, X\}$ *is an IGMRA for* $L^2(\mathbb{R}^d)$*, with* $\mathcal{V} = \{\mathcal{B}_{U_j}\}_{j\in\mathbb{Z}}$*, and* $X = \{X^j\}_{j\in\mathbb{Z}}$*.* $\{\varphi_j\}_{j\in\mathbb{Z}}$ *is the generating family of IGMRA.*

Proof We begin by showing that the subspaces $\mathcal{B}_{U_j}$ verify the properties *(a), (b), (c), and (d)* of Definition 4.1:

(a) $U_j \subset U_{j+1}\ \forall j \Rightarrow \overline{U_j} \subset \overline{U_{j+1}}\ \forall j$, then $\mathcal{B}_{U_j} \subset \mathcal{B}_{U_{j+1}}\ \forall j$. Therefore:

$$...\mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset ...$$

(b) We will show that $\overline{\cup_{j\in\mathbb{Z}}\mathcal{V}_j} = L^2(\mathbb{R}^d)$.
The set $D := \{f \in L^2(\mathbb{R}^d) :\ \text{supp}\ \widehat{f}$ is compact$\}$ is dense in $L^2(\mathbb{R}^d)$.
If $f \in D$, $S :=$ supp $\widehat{f}$ is compact, then as $\{U_j\}_{j\in\mathbb{Z}}$ is a covering by open sets of $\mathbb{R}^d$, there exists a set $\{j_1, j_2, \ldots, j_n\} \subset \mathbb{Z}$ so that $S \subset \cup_{i=1}^n U_{j_i}$. By hypothesis *1)* $S \subset U_{j_n}$, then supp $\widehat{f} \subset \overline{U_{j_n}}$, therefore $f \in \mathcal{B}_{U_{j_n}} = \mathcal{V}_{j_n}$. Therefore, $D \subset \cup_{j\in\mathbb{Z}}\mathcal{V}_j$, and this implies that $\overline{\cup_{j\in\mathbb{Z}}\mathcal{V}_j} = L^2(\mathbb{R}^d)$.
(c) We can see that $\cap_{j\in\mathbb{Z}}\mathcal{V}_j = 0$.
If $f \in \cap_{j\in\mathbb{Z}}\mathcal{V}_j$ then $S :=$ supp $\widehat{f} \subset \overline{U_j} = U_j \cup \partial U_j\ \forall j$. Then $S \subset \cap_{j\in\mathbb{Z}}(U_j \cup \partial U_j) = (\cap_{j\in\mathbb{Z}}U_j) \cup (\cap_{j\in\mathbb{Z}}\partial U_j)$. By hypothesis, both sets of the union have a null measurement, then $\mu(S) = 0$, thus $\widehat{f} = 0\ a.e$, and consequently $f = 0\ a.e$.
(d) We can now prove that $\{\widehat{\varphi_j}\ e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$ is an outer frame for $\mathcal{K}_{U_j}$. By hypothesis $b)$ the set $\{e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{K}_{U_{\varepsilon_j}}$. Furthermore, $U_j \subset U_{\varepsilon_j}$ and $\mu(\partial U_j) = 0$, then

$$\{(e_{x_k^j}\chi_{U_{\varepsilon_j}})_{|U_j}\}_{k\in\mathbb{Z}} = \{e_{x_k^j}\chi_{U_j}\}_{k\in\mathbb{Z}}$$

is a frame for $\mathcal{K}_{U_j}$ (c) of Lemma 3.6). Because $0 < c_j \le |\widehat{\varphi_j}(\omega)| \le C_j < \infty\ \forall\omega \in U_j$, by c) of the Proposition 3.8, it follows that $\{\widehat{\varphi_j}\ e_{x_k^j}\chi_{U_j}\}_{k\in\mathbb{Z}}$ is a frame of $\mathcal{K}_{U_j}$. Then, by definition of outer frames, we obtain

$$\{\widehat{\varphi_j}\ e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$$

is an outer frame of $\mathcal{K}_{U_j}$. Then, by Lemma 3.6(b))

$$\{T_{x_k^j}\varphi_j\}_{k\in\mathbb{Z}} = \{\varphi_j(x - x_k^j)\}_{k\in\mathbb{Z}}$$

is an outer frame of $\mathcal{V}_j = \mathcal{B}_{U_j}$

As shown in (a), (b), (c) and (d), $\{\mathcal{V}, X\}$ is an IGMRA of $L^2(\mathbb{R}^d)$.Let $\mathcal{V} = \{\mathcal{B}_{U_j}\}_{j\in\mathbb{Z}}$, $X = \{X^j\}_{j\in\mathbb{Z}}$, and $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the IGMRA generating family. □

[2] Since U_{ε_j} is open and bounded, $r_j > 0$ exists, so that $U_{\varepsilon_j} \subset B_{r_j}(0)$. We chose a separate set of points X^j, for each $j \in \mathbb{Z}$, so that its gap ρ_j verifies $r_j.\rho_j < 1/4$, then by the Beurling Theorem, $\{e_{x_k^j}\chi_{B_{r_j}(0)}\}_{k\in\mathbb{Z}}$, is a frame for $\mathcal{K}_{B_{r_j}(0)}$. As a $\mu(\partial U_{\varepsilon_j}) = 0$, by c) of Lemma 2, we have that $\{e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$ is frame of U_{ε_j}.

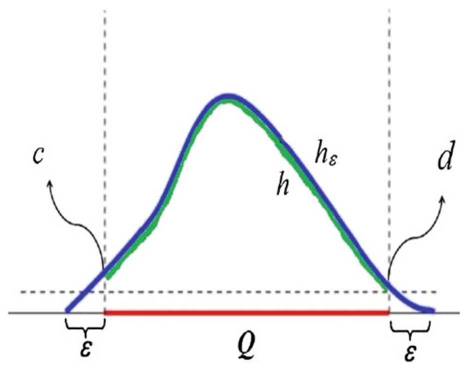

Fig. 1 Example in $L^2(\mathbb{R})$. Supp $h = \overline{Q}$, $\mu(\partial Q) = 0$, supp $h_\varepsilon = \overline{Q}_\varepsilon = \overline{\{x \in \mathbb{R} : d(x, Q) < \varepsilon\}}$, $Q_\varepsilon \supset Q$, h_ε is a continuous function, and $h \neq 0$ in Q

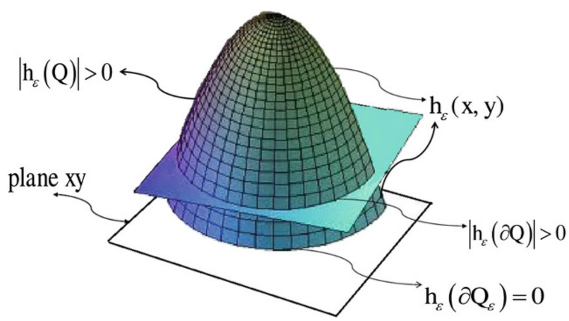

Fig. 2 Example in $L^2(\mathbb{R}^2)$. Supp $h_\varepsilon = \overline{Q}_\varepsilon$, h_ε is a continuous function, $Q_\varepsilon \supset Q$, $\mu(\partial Q) = 0$, $h_{\varepsilon|_Q} = h$, and $|\, h \,| \neq 0$ in Q

In Theorem 4.9, it is proved that we can associate this type of multiresolution analysis frames of translations with good localization, in the sense of Definition 4.5.

The central idea of this approach, is basically the same used by Ingrid Daubechies and Ron DeVore [15]. Given a bounded set $Q \subset \mathbb{R}^d$ with $\mu(\partial Q) = 0$, and a function h with supp $h = \overline{Q}$ so that $0 < c \leq |h| \leq d < \infty$, we consider a set $Q_\varepsilon \supset Q$ (Q_ε also bounded), and a function h_ε of class $\mathcal{C}^n$ so that supp $h_\varepsilon = \overline{Q_\varepsilon}$ and $h_{\varepsilon|_Q} = h$ (see Figs. 1, 2). From these results, we worked with the function $\check{h}_\varepsilon$ that has good localization because $h_\varepsilon \in \mathcal{C}^n$, thus obtaining frames with good localization.

Remark 4.7 We observe that,

$$\langle \widehat{f}, h_\varepsilon \rangle = \langle \widehat{f}, h \rangle \;\forall \widehat{f} \in \mathcal{K}_Q$$

then

$$\langle f, \check{h}_\varepsilon \rangle = \langle f, \check{h} \rangle \;\forall f \in \mathcal{B}_Q$$

We need to incorporate a definition, contained in [1] before exposing the following theorem.

Definition 4.8 A set $\mathcal{H} := \{h_j\}_{j \in J}$ of measurable functions on $\mathbb{R}^d$ is called a Riesz partition of unity (RPU), if there exist constants $0 < p \leq P < +\infty$ such that $p \leq \sum_{j \in J} |h_j(x)|^2 \leq P$ $a.e\ x \in \mathbb{R}^d$.
Let $S = \{S_j\}_{j \in J}$ be a family of measurable subsets of $\mathbb{R}^d$. A *Riesz partition of unity associated to* S, is a set $H := \{h_j\}_{j \in J}$ of measurable functions, such that

- Supp $h_j \subseteq S_j$
- There exist constants $0 < p \leq P < +\infty$ such that

$$p \leq \sum_{j \in J} |h_j(x)|^2 \leq P \qquad a.e\ x \in \bigcup_{j \in J} S_j$$

Theorem 4.9 *Under the same hypothesis as in Theorem 2 for $\{U_j\}_{j\in\mathbb{Z}}$, the sequence $\{\varepsilon_j\}_{j\in\mathbb{Z}}$, $\{U_{\varepsilon_j}\}_{j\in\mathbb{Z}}$ and the sets X^j. If:*

1. $0 < m := \inf_j m_j \leq M := sup_j\ M_j < \infty$, *with m_j and M_j bounds of the frame $\{e_{x_k^j}\chi_{U_{\varepsilon_j}}\}$ of $\mathcal{K}_{U_{\varepsilon_j}}$,*
2. *We define:*
   - $Q_j := U_{j+1}\backslash U_j$, *and*
   - $Q_j^{\varepsilon_{j+1}} := \{x \in \mathbb{R}^d : d(x, Q_j) < \varepsilon_{j+1}\}$
3. *We assume set of functions $\{h_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ so that:*
   (a) *$\{h_j\}_{j\in\mathbb{Z}}$ is Riesz partition of unity with bounds p and P,*
   (b) $h_j \in \mathcal{K}_{Q_j^{\varepsilon_{j+1}}}\ \forall j$.
   (c) $0 < d_j \leq |h_j(\omega)|^2 \leq D_j < \infty\ \forall\omega \in Q_j$
   Then,

$$\{\check{h}_j(x - x_k^{j+1})\}_{j\in\mathbb{Z},k\in\mathbb{Z}} \tag{2}$$

is a frame of translations of $L^2(\mathbb{R}^d)$, with bounds mp and MP; and the pair $\{\check{h}_j, X^{j+1}\}_{j\in\mathbb{Z}}$ is a generalized wavelet associated with the IGMRA $\{\mathcal{V}, X\}$ determined in Theorem 4.6.

Before proving this theorem, we will introduce Theorem 4.10 and its remark of Aldroubi et al. [1].

Theorem 4.10 (Aldroubi, Cabrelli, Molter) *Let J and K be countable index sets. If the set $\mathcal{H} = \{h_j\}_{j\in J} \subset L^2(\mathbb{R}^d)$ is RPU, with bounds p and P. For each $j \in J$, we define*

(a) $W_j := \overline{\{\overline{h_j} f\ :\ f \in L^2(\mathbb{R}^d)\}}$.
(b) *Let* $0 < c \leq p$,
$Q_j := Q_j(c) = \{x \in \mathbb{R}^d\ :\ |h_j(x)|^2 > c\}$, *for each* $j \in J$.

For a given c, we discard all those j such that Q_j has measure zero. Note that if $J_0 = \{j \in J : \mu(Q_j) > 0\}$, then we can only claim that $\mathcal{H}_0 = \{h_j\ :\ j \in J_0\}$ is a RPU associated to $\{Q_j\}_{j\in J_0}$ with bounds c and P.

(1) *Let $\{g_{jk}\}_{k\in K}$ be a frame for W_j with bounds m_j and M_j, respectively. If $m := \inf_j m_j > 0$ and $M := sup_j\ M_j < \infty$, then the set $\{h_j g_{jk}\}_{j\in J,k\in K}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds $p.m$ and PM.*
(2) *Let $0 < c \leq p$ be, if for each $j \in J_0$, $\{g_{jk}\}_{k\in K}$ is a frame for $\mathcal{K}_{Q_j}$ with bounds m_j and M_j, respectively, and it is verified that $m := \inf_j m_j > 0$ and $M := sup_j\ M_j < \infty$, then $\{h_j g_{jk}\}_{j\in J_0,k\in K}$ is a frame for $\mathcal{K}_{\cup Q_j}$, with bounds $c.m$ and $P.M$.*

Remark 4.11 (of Theorem 4.10)

- Note that in the previous theorem, instead of choosing a frame for the subspaces W_j, we could have chosen any collection of functions of $L^2(\mathbb{R}^d)$ that form an outer frame for W_j.
- If h is a bounded function and $Q = \operatorname{supp} h$, then it is easy to see that $closure_{L^2}\{hf\ :\ f \in L^2(\mathbb{R}^d)\} = \mathcal{K}_Q$ if and only if $\mu(Q) = \mu(\{x\ :\ |h(x)| > 0\})$. So, the spaces W_j will coincide in most of the cases with $\mathcal{K}_{\operatorname{supp}(h_j)}$.

Proof of Theorem 4.9 The subspaces $\mathcal{W}_j := \mathcal{B}_{Q_j}$ are the orthogonal complements of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}\ \forall J \in \mathbb{Z}$. We will prove that $\{h_j e_{x_k^{j+1}}\chi_{Q_j^{\varepsilon_{j+1}}}\}_{k\in\mathbb{Z}}$ is an outer frame for $\mathcal{K}_{Q_j}$.

The set $\{e_{x_k^{j+1}}\chi_{U_{\varepsilon_{j+1}}}\}$ is frame of $\mathcal{K}_{U_{\varepsilon_{j+1}}}$ with bounds m_{j+1} and M_{j+1}, $Q_j \subset U_{\varepsilon_{j+1}}$ and $\mu(\partial Q_j) = 0$, then by (c) of Lemma 3.6, the collection $\{e_{x_k^{j+1}}\chi_{Q_j}\}_{k\in\mathbb{Z}}$ is frame for $\mathcal{K}_{Q_j}$ with the same bounds. Then by hypothesis (c) of the Proposition 3.8, we obtain:

$$\{h_j e_{x_k^{j+1}}\chi_{Q_j}\}_{k\in\mathbb{Z}} \tag{3}$$

is frame for $\mathcal{K}_{Q_j}$. We will prove that its bounds are $d_j.m_{j+1}$, and $D_j.M_{j+1}$.

Let $f \in \mathcal{K}_{Q_j}$ then $\overline{h}_j f \in \mathcal{K}_{Q_j}$. As $\mathcal{K}_{Q_j} \subset \mathcal{K}_{U_{\varepsilon_{j+1}}}$ and $\{e_{x_k^{j+1}}\chi_{U_{\varepsilon_{j+1}}}\}$ is frame of $\mathcal{K}_{U_{\varepsilon_{j+1}}}$ with bounds m_{j+1} and M_{j+1} then

$$m_{j+1}\|\overline{h}_j f\|^2 \leq \sum_{k\in\mathbb{Z}} | < \overline{h} f, e_{x_k^{j+1}}\chi_{U_{\varepsilon_{j+1}}} > | \leq M_{j+1}\|\overline{h}_j f\|^2 \tag{4}$$

Since $\overline{h}_j f \in \mathcal{K}_{Q_j}$ then

$$\sum_{k\in\mathbb{Z}} | < \overline{h} f, e_{x_k^{j+1}} \chi_{U_{\varepsilon_{j+1}}} > | = \sum_{k\in\mathbb{Z}} | < \overline{h} f, e_{x_k^{j+1}} \chi_{Q_j} > |$$

Furthermore by hypothesis 3.c

1. $d_j m_{j+1} \|f\|^2 \leq m_{j+1} \|\overline{h}_j f\|^2$, and
2. $M_{j+1} \|\overline{h}_j f\|^2 \leq D_j M_{j+1} \|f\|^2$

Therefore of (4) it follows that

$$d_j m_{j+1} \|f\|^2 \leq \sum_{k\in\mathbb{Z}} | < f, h_j e_{x_k^{j+1}} \chi_{Q_j} > | \leq D_j M_{j+1} \|f\|^2 \tag{5}$$

We can now prove that $\{h_j e_{x_k^{j+1}} \chi_{Q_j^{\varepsilon_{j+1}}}\}_{k\in\mathbb{Z}}$ is outer frame for $\mathcal{K}_{Q_j}$ with bounds $d_j m_{j+1}$ and $D_j M_{j+1}$. Let $f \in \mathcal{K}_{Q_j}$ be, then

$$d_j m_{j+1} \|f\|^2 \leq \sum_{k\in\mathbb{Z}} | < f, h_j e_{x_k^{j+1}} \chi_{Q_j} > |^2 \leq D_j M_{j+1} \|f\|^2 \tag{6}$$

since supp $f \subset \overline{Q_j}$ and $\mu(\partial Q_j) = 0$, we have:

$$< f, h_j e_{x_k^{j+1}} \chi_{Q_j} > = < f, h_j e_{x_k^{j+1}} \chi_{Q_j^{\varepsilon_{j+1}}} > \tag{7}$$

then the expression (6) is identical to:

$$d_j m_{j+1} \|f\|^2 \leq \sum_{k\in\mathbb{Z}} | < f, h_j e_{x_k^{j+1}} \chi_{Q_j^{\varepsilon_{j+1}}} > |^2 \leq D_j M_{j+1} \|f\|^2 \tag{8}$$

from which we deduce that $\{h_j e_{x_k^{j+1}} \chi_{Q_j^{\varepsilon_{j+1}}}\}_{k\in\mathbb{Z}}$ is outer frame for $\mathcal{K}_{Q_j}$ with bounds $d_j m_{j+1}$, and $D_j M_{j+1}$. By b) of Lemma 3.6, the set

$$\{\check{h}_j(x - x_k^{j+1})\}_{k\in\mathbb{Z}} \tag{9}$$

is a outer frame for $\mathcal{B}_{Q_j} = \mathcal{W}_j$ with bounds $d_j m_{j+1}$ and $D_j M_{j+1}$.

We will prove that $\{\check{h}_j(x - x_k^{j+1})\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is frame of $L^2(\mathbb{R}^d)$:
$h_j \in \mathcal{K}_{Q_j^{\varepsilon_{j+1}}} \Rightarrow W_j \subset \mathcal{K}_{Q_j^{\varepsilon_{j+1}}} \subset \mathcal{K}_{U_{\varepsilon_{j+1}}}$, where W_j are the subspaces defined in Theorem 4.10. The $\{e_{x_k^{j+1}} \chi_{U_{\varepsilon_{j+1}}}\}_{k\in\mathbb{Z}}$ set is frame of $\mathcal{K}_{U_{\varepsilon_{j+1}}}$ with bounds m_j and M_j, then $\{e_{x_k^{j+1}} \chi_{U_{\varepsilon_{j+1}}}\}_{k\in\mathbb{Z}}$ is a outer frame of W_j with bounds m_j and M_j. Since $\{h_j\}_{j\in\mathbb{Z}}$ is RPU with bounds p and P by Theorem 4.10 (part 1)) and its subsequent Remark, the set:[3]

$$\{h_j e_{x_k^{j+1}} \chi_{U_{\varepsilon_{j+1}}}\}_{j\in\mathbb{Z},k\in\mathbb{Z}} = \{h_j e_{x_k^{j+1}} \chi_{Q_j^{\varepsilon_{j+1}}}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

is frame of $L^2(\mathbb{R}^d)$ with bounds pm and PM. Then,

$$\{\check{h}_j(\cdot - x_k^{j+1})\}_{j\in\mathbb{Z},k\in Z}$$

is frame for $L^2(\mathbb{R}^d)$ with bounds pm and PM. □

Remark 4.12 1. $\{\check{h}_j, X^{j+1}\}_{j\in\mathbb{Z}}$ is a generalized wavelet (frame of translations) associated to IGMRA $\{\mathcal{V}, X\}$ determined in Theorem 4.6.
2. The frame $\{\check{h}_j(\cdot - x_k^{j+1})\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ may have a localization as good as desired, depending on the degree of smoothness of the functions h_j.

[3] $h_j \in \mathcal{K}_{Q_j^{\varepsilon_{j+1}}}$ and $\mu(\partial Q_j^{\varepsilon_{j+1}}) = 0$.

5 Examples of IGMRA and associated wavelets in $L^2(\mathbb{R})$

Let $Q = (-1, -1/2] \cup [1/2, 1), 0 < \varepsilon < 1/2, \widehat{h}_+$ be, a real function so that supp $\widehat{h}_+ \subseteq Q_\varepsilon^+ = [1/2 - \varepsilon, 1 + \varepsilon]$, and $0 < c \leq |\widehat{h}_+| \leq C < \infty$ on $[1/2, 1]$.

We define:

1. $\widehat{h}(\xi) := \begin{cases} \widehat{h}_+(\xi) & \text{if} \quad \xi \in Q_\varepsilon^+ \\ \widehat{h}_+(-\xi) & \text{if} \quad \xi \in -Q_\varepsilon^+ \end{cases}$
2. $\widehat{h}_j := \widehat{h}(2^{-j})$.

Let:

- $U_j := (-2^j, 2^j)$
- $U_{\varepsilon_j} := 2^j(-1-\varepsilon, 1+\varepsilon), \varepsilon_j := 2^j \varepsilon \; \forall \; j)$
- $X^j := \{x_k^j\}_{k \in \mathbb{Z}}$ separate set of $\mathbb{R}$, which verifies that $\rho(X^j) 2^j (1+\varepsilon) < 1/4$.

If $0 < m := \inf_j m_j \leq M := sup_j \, M_j < \infty$ for m_j and M_j bounds of the frame $\{e_{x_k^j} \chi_{U_{\varepsilon_j}}\}_{k \in \mathbb{Z}}$ of $\mathcal{K}_{U_{\varepsilon_j}}$, then:

(a) $\{\mathcal{V}, X\}$ is an IGMRA, where $\mathcal{V} = \{\mathcal{B}_{U_j}\}_{j \in \mathbb{Z}}$, $X = \{X^j\}_{j \in \mathbb{Z}}$, and $\{\check{\chi}_{U_{\varepsilon_j}}\}_{j \in \mathbb{Z}}$ a family of generators.

(b) The pair $\{h_j, X^{j+1}\}_{j \in \mathbb{Z}}$ is a generalized wavelet for $L^2(\mathbb{R})$ associated with IGMRA $\{\mathcal{V}, X\}$.

Before the proof, the previous example adds a definition given in [1].

Definition 5.1 Let $\boldsymbol{S} = \{S_j\}_{j \in J} \subseteq \mathbb{R}^d$ be a family of measurable subsets such that $\boldsymbol{S}$ is a covering of $\mathbb{R}^d$. Define

$$\rho_{\boldsymbol{S}}(x) := \#\{j \in J : x \in S_j\} = \sum_{j \in J} \chi_{S_j}(x)$$

where $\#(B)$ is the cardinal of the set B. *The covering index of* $\boldsymbol{S}$ is defined as $\rho_{\boldsymbol{S}} =: \|\rho_{\boldsymbol{S}}\|_\infty$.

Proof (a) We see the hypotheses of Theorem 4.6 are verified,

- $U_j = (-2^j, 2^j) \subset (-2^{j+1}, 2^{j+1}) = U_{j+1}$ for all $j \in \mathbb{Z}$.
- $\cap_{j \in \mathbb{Z}} U_j = 0$, then $\mu(\cap_{j \in \mathbb{Z}} U_j) = 0$
- $\mu(\partial(U_j)) = 0 \; \forall j$
- $\cup_{j \in \mathbb{Z}} U_j = \mathbb{R}$

Since $\rho(X^j).2^{j+1}(1+\varepsilon) < 1/4$ then by Theorem 3.4, the set $\{e_{x_k^j} \chi_{U_{\varepsilon_j}}\}_{k \in \mathbb{Z}}$ is a frame of $\mathcal{K}_{U_{\varepsilon_j}}$. If we take $\widehat{\varphi}_j = \chi_{U_{\varepsilon_j}}$, all the hypotheses of Theorem 4.6 are fulfilled, with which $\{\mathcal{V}, X\}$ is a IGMRA.

(b) Defining,

- $Q_{j-1} := U_j \backslash U_{j-1} = 2^j Q$
- $Q_{j-1}^{\varepsilon_j} := \{x \in \mathbb{R} : d(x, Q_{j-1}) \leq \varepsilon_j = 2^j \varepsilon\}$

We observe that:

- supp $\widehat{h} \subseteq -Q_\varepsilon^+ \cup Q_\varepsilon^+$
- supp $\widehat{h}_j \subseteq Q_{j-1}^{\varepsilon_j}$

Let:

- $\mathcal{Q} := \{Q_j\}_{j \in \mathbb{Z}}$
- $\mathcal{Q}_\varepsilon := \{Q_j^{\varepsilon_{j+1}}\}_{j \in \mathbb{Z}}$

$\mathcal{Q}$ is a covering of $\mathbb{R} \backslash \{0\}$ by disjoint sets.

Because $Q_j \subset Q_j^{\varepsilon_{j+1}}$, $\mathcal{Q}_\varepsilon$ is also a covering of $\mathbb{R} \backslash \{0\}$.

The covering index of $\mathcal{Q}_\varepsilon$ is also finite and grows with ε. We will prove this.

Let $x \in \mathbb{R}$ and $0 < \varepsilon < 1/2$. Assume that $0 < j_0 \in \mathbb{Z}$ so that $x \in Q_{j_0}$[4].

If $k = 0$, since $x \in Q_{j_0}$ and $Q_{j_0} \subset Q_{j_0}^{\varepsilon_{j_0+1}}$ then $x \in Q_{j_0}^{\varepsilon_{j_0+1}}$.

Let $k \neq 0$, then

[4] Similarly is shown that the covering index of $\mathcal{Q}_\varepsilon$ is finite and grows with ε if $j_0 \leq 0$.

– If $k > 0$:

$$2^{j_0+k} - 2^{j_0+1} \leq d(x, Q_{j_0+k}) \leq 2^{j_0+k} - 2^{j_0} \tag{10}$$

– If $k < 0$:

$$2^{j_0} - 2^{j_0+k+1} \leq d(x, Q_{j_0+k}) \leq 2^{j_0+1} - 2^{j_0+k+1} \tag{11}$$

It follows
1. If $k > 0$ and $x \in Q_{j_0+k}^{\varepsilon_{j_0+k+1}}$, there exists a constant $0 < k_0(\varepsilon) \in \mathbb{Z}$ so that $k \leq k_0(\varepsilon)$.
2. If $k < 0$ and $x \in Q_{j_0+k}^{\varepsilon_{j_0+k+1}}$, then $k = -1$.

We will prove this.
1. Assume $k > 0$. If $x \in Q_{j_0+k}^{\varepsilon_{j_0+k+1}}$, then
   $d(x, Q_{j_0+k}) \leq 2^{j_0+k+1}\varepsilon$. From (10), it follows that

$$2^{j_0+k} - 2^{j_0+1} \leq 2^{j_0+k+1}\varepsilon$$

   or

$$2^{j_0+k} - 2^{j_0} \leq 2^{j_0+k+1}\varepsilon$$

   Therefore,

$$1/2 - 1/2^k \leq \varepsilon \quad \text{or} \quad 1/2 - 1/2^{k+1} \leq \varepsilon$$

   This is only valid for a finite number of k ($0 < \varepsilon < 1/2$). Then, there exists
   $k_0(\varepsilon) := max\{k > 0 : \ 1/2 - 1/2^k \leq \varepsilon\}$. So $k \leq k_0(\varepsilon)$.
1. Suppose now $k < 0$. Since $x \in Q_{j_0+k}^{\varepsilon_{j_0+k+1}}$, from (11)

$$2^{j_0} - 2^{j_0+k+1} \leq 2^{j_0+k+1}\varepsilon$$

   or

$$2^{j_0+1} - 2^{j_0+k+1} \leq 2^{j_0+k+1}\varepsilon$$

   Therefore,

$$1/2^{k+1} - 1 \leq \varepsilon \quad \text{or} \quad 1/2^k - 1 \leq \varepsilon$$

   Since $1/2^k - 1 \geq 1 \ \forall k < 0$ there is no $k < 0$ such that $1/2^k - 1 \leq \varepsilon$.
   The expression $1/2^{k+1} - 1 < \varepsilon$ is only valid for $k = -1$.

Therefore, $\rho_{\mathcal{Q}_\varepsilon}(x) \leq k_0(\varepsilon) + 2 \quad \forall x \in \mathbb{R}$.

Remark 5.2 We observed that
– $\rho_{\mathcal{Q}_\varepsilon}(x)$ does not depend on j_0.
– $k_0(\varepsilon)$ increases if ε increases.

Then, the index of covering of $\mathcal{Q}_\varepsilon$ is

$$\rho_{\mathcal{Q}_\varepsilon} \leq k_0(\varepsilon) + 2$$

Now let us ascertain that hypotheses of Theorem 4.9 are verified.
We observe that $\{\widehat{h}_j\}_{j\in\mathbb{Z}}$ is a Riesz partition of unity,
(i) If $\xi \in \mathbb{R}$, then $\rho_{\mathcal{Q}_\varepsilon}(x) \leq \rho_{\mathcal{Q}_\varepsilon}$ and since $|\widehat{h}_j| \leq C$, then:

$$\sum_{j\in\mathbb{Z}} |\widehat{h}_j(\xi)|^2 \leq C^2 \rho_{\mathcal{Q}_\varepsilon}$$

(ii) $\forall \xi \in \mathbb{R}$ there is a single $j \in \mathbb{Z}$ such as $2^{-j}\xi \in Q$, then $|\widehat{h}_j(\xi)|^2 = |\widehat{h}(2^{-j}\xi)|^2 > c^2$

Therefore,

$$c^2 \leq \sum_{j\in\mathbb{Z}} |\widehat{h}_j(\xi)|^2 \leq C^2 \rho_{\mathcal{Q}_\varepsilon} \quad a.e\ \xi \in \mathbb{R}^d$$

By definition of $\widehat{h}_j$, the hypothesis about h^+, and since supp $\widehat{h}_j \subseteq Q_{j-1}^{\varepsilon_j}$, we have:

$$0 < c \leq |\widehat{h}_j(\xi)| \leq C < \infty \quad \forall \xi \in Q_j$$

Then, according to the hypothesis of Theorem 4.9,

$$\{h_j(\cdot - x_k^{j+1})\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

is a frame of $L^2(\mathbb{R})$ with bounds mc^2 and $MC^2\rho_{\mathcal{Q}_\varepsilon}$, and $\{h_j, X^{j+1}\}_{j\in\mathbb{Z}}$ is a generalized wavelet associated with IGMRA $\{\mathcal{V}, X\}$.

As $h_j = 2^j h(2^j \cdot)$ and $h = 2Re\ h_+$, then the previous frame is:

$$\{2^{j+1} Re\ h_+(2^j(\cdot - x_k^j))\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

□

Based on these results, we present concrete examples.

5.1 Wavelets associated to the IGMRA determined in Sect. 5

5.1.1 Wavelets without good localization

Let $\varepsilon = 0$, $\widehat{h}_+ = \chi_{[1/2,1]}$, and the sets X^j so that $\rho(X^j) < 1/2^{j+2}$; since $Re\ h_+(x) = 1/2\text{sinc}\left(\frac{\pi}{2}x\right)\cos\left(\frac{3}{2}\pi x\right)$, then

$$\{2^j \text{sinc}(2^{j-1}\pi(x - x_k^j))\cos\left(2^{j-1}3\pi\left(x - x_k^j\right)\right)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

is a generalized wavelet of $L^2(\mathbb{R})$ associated with IGMRA $\{\mathcal{V}, X\}$ of the Sect. 5. This frame does not have good localization because its functions decay as $O(|x|^{-1})$.

5.1.2 Wavelets with good localization

For the same IGMRA, we take $\varepsilon = 1/4$, and $\widehat{h}_+ = n\beta_{n-1}((\xi - 1/4)n)$, being β_{n-1} the spline function of degree n whose support is $[0, n]$. [5]

If $\widehat{h}(\xi) := \begin{cases} \widehat{h}_+(\xi) & \text{if} \quad \xi \in Q_\varepsilon^+ := [1/4, 5/4] \\ \widehat{h}_+(-\xi) & \text{if} \quad \xi \in -Q_\varepsilon^+ \end{cases}$ then supp $\widehat{h} = [-5/4, -1/4] \cup [1/4, 5/4]$ and

i. $0 < c \leq |\widehat{h}|$ on $Q := [-1, -1/2] \cup [1/2, 1]$ with $c = n\beta_{n-1}((1/4)n) = n\beta_{n-1}((3/4)n)$
ii. By definition $|\widehat{h}| \leq n$.

Let $Q_j = 2^j Q$ and $\widehat{h}_j = \widehat{h}(2^{-j})$, if $\varepsilon_j = 2^j \varepsilon = 2^{j-2}$ and

$$Q_{j-1}^{\varepsilon_j} := \{x \in \mathbb{R} : d(x, 2^{j-1}Q) \leq 2^{j-2}\}$$

then the index of covering $\rho_{\mathcal{Q}_\varepsilon}$ of $\mathcal{Q}_\varepsilon := \{Q_{j-1}^{\varepsilon_j}\}_{j\in\mathbb{Z}}$ is finite $(0 < \varepsilon < 1/2)$.

We deduce that:

[5] The term β-spline is due to Isaac Jacob Schoenberg, Romanian mathematician (April 21, 1903–February 21, 1990) and is short of *basic spline*. β_0 is defined as $\chi_{[0,1]}$, and β_k recursively as the convolution product $\beta_k = \beta_{k-1} * \beta_o$ whose support is $[0, k+1]$.

1. $\text{supp}\,\widehat{h}_j = Q_j^{\varepsilon_{j+1}}$
2. $0 < c \leq |\widehat{h}_j(\xi)| \leq n \quad \forall \xi \in Q_j$
3. $c^2 \leq \sum_{j\in\mathbb{Z}} |\widehat{h}_j(\xi)|^2 \leq n^2 \rho_{\mathcal{Q}_\varepsilon} \quad a.e\ \xi \in \mathbb{R}^d$

Again the hypotheses of Theorem 4.9 are fulfilled. Since $Re\ h_+(x) = \cos(\frac{3}{2}\pi x)\text{sinc}^n(\frac{\pi}{n}x)$, we obtain a generalized wavelet associated with IGMRA $\{\mathcal{V}, X\}$:

$$\left\{2^{j+1}\cos(2^{j-1}3\pi(x - x_k^j))\text{sinc}^n\left(2^j\ \frac{\pi}{n}(x - x_k^j)\right)\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

This frame has good localization. Its functions decay as $O(|x^{-n}|)$, because $\beta_{n-1} \in C^{n-1}$.

6 Examples of IGMRA and associated wavelets in $L^2(\mathbb{R}^2)$

Let $\varepsilon_j = 4^j\varepsilon\ \forall j \in \mathbb{Z}$ with $0 < \varepsilon < 1/4$, and the sets:

1. $U_j = B_{4^j}(0) \quad \forall j \in \mathbb{Z}$
2. $U_{\varepsilon_j} = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, U_j) < \varepsilon_j\} = B_{4^j(1+\varepsilon)}(0)\ \forall j$
3. $Q_j = U_{j+1}\backslash U_j = \{(u, v) \in \mathbb{R}^2 : 4^j \leq u^2 + v^2 < 4^{j+1}\} \quad \forall j \in \mathbb{Z}$
4. $Q_j^{\varepsilon_{j+1}} = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, Q_j) < \varepsilon_{j+1}\}\ \ \forall j \in \mathbb{Z}$
5. $X^j = \{x_k^j\}_{k\in\mathbb{Z}} \subset \mathbb{R}^2$ a separate set such that $\forall j \in \mathbb{Z}$, the set $\{e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$ is frame of $\mathcal{K}_{U_{\varepsilon_j}}$ with bounds m_j and M_j.

Then, $\{\mathcal{V}, X\}$ is an IGMRA of $L^2(\mathbb{R}^2)$, where $\mathcal{V} = \{\mathcal{B}_{U_j}\}$, $X = \{X^j\}_{j\in\mathbb{Z}}$ and $\{\check{\chi}_{U_{\varepsilon_j}}\}_{j\in\mathbb{Z}}$ is a generating family of IGMRA. The proof of this result is omitted because it is similar to the demonstration given for the example of Sect. 5.

6.1 Wavelets associated to the IGMRA determined in Sect. 6

If $0 < m := \inf_j m_j \leq M := sup_j\ M_j < \infty$ for m_j and M_j lower and upper bounds of the frame $\{e_{x_k^j}\chi_{U_{\varepsilon_j}}\}_{k\in\mathbb{Z}}$ of $\mathcal{K}_{U_{\varepsilon_j}}$, there are different generalized wavelets associated with IGMRA $\{\mathcal{V}, X\}$ of the Sect. 6. If we take a set of functions $\{\widehat{h}_j\}_{j\in\mathbb{Z}}$ so that:

(i) $\widehat{h}_j \in \mathcal{K}_{Q_j^{\varepsilon_{j+1}}}\ \forall j$

(ii) $0 < c \leq |\widehat{h}_j| \leq k < \infty$ on Q_j.

Then, by Theorem 4.9, $\{T_{x_k^j}h_j\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a generalized wavelet for $L^2(\mathbb{R}^2)$ associated with IGMRA.

6.1.1 Wavelets without good localization

Let $\varepsilon = 0$ and $\widehat{h}_j = \chi_{Q_j}$, the hypotheses of Theorem 4.9 are verified, then:

$$\{T_{x_k^j}h_j\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

will be a generalized wavelet associated with the IGMRA $\{\mathcal{V}, X\}$ of the Sect. 6. This frame does not have good localization because $\chi_{Q_j^{\varepsilon_{j+1}}}$ are not continuous.

6.1.2 Wavelets with good localization

If $0 < \varepsilon < 1/4$

$$Q_j^{\varepsilon_{j+1}} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : 4^j(1 - 4\varepsilon) < \xi_1^2 + \xi_2^2 < 4^{j+1}(1 + \varepsilon)\}$$

and we take

$$\widehat{h}_j(\xi_1, \xi_2) = n\beta_{n-1}\left(\left(\xi_1^2 + \xi_2^2 - 4^j(1-4\varepsilon)\right)\frac{n}{4^j(3+8\varepsilon)}\right)$$

It follows:

– supp $\widehat{h}_j = \overline{Q_j^{\varepsilon_{j+1}}}$ $\forall j$, then $\widehat{h}_j \in \mathcal{K}_{Q_j^{\varepsilon_{j+1}}}$
– $|\widehat{h}_j| \leq n$ and $0 < c \leq |\widehat{h}_j|$ on Q_j ($c = \widehat{h}_j(\widetilde{\xi}_1, \widetilde{\xi}_2)$, with $\widetilde{\xi}_1^2 + \widetilde{\xi}_2^2 = 4^j$ o $\widetilde{\xi}_1^2 + \widetilde{\xi}_2^2 = 4^{j+1}$)

The functions $\{\widehat{h}_j\}_{j\in\mathbb{Z}}$ form a RPU with bounds $p = c^2$ and $P = \rho_{\mathcal{Q}_\varepsilon} n$, where $\rho_{\mathcal{Q}_\varepsilon}$ is the covering index of $\mathcal{Q}_\varepsilon := \{Q_j^{\varepsilon_{j+1}}\}_{j\in\mathbb{Z}}$, it is finite because $\varepsilon < 1/4$ [6]. Theorem 4.9 hypotheses are verified, then the generalized wavelet:

$$\{h_j(x - x_k^j)\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

will be associated with the same IGMRA $\{\mathcal{V}, X\}$ of the Sect. 6. This frame is good localized because it decays as $O(|x|^{-n})$, due to $\widehat{h}_j$ is $\mathcal{C}^{n-1}$ class. Its bounds will be $c^2 m$ and $\rho_{\mathcal{Q}_\varepsilon} nM$.

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[6] We do not include this demonstration, this is similar to that given in the proof of IGMRA of $L^2(\mathbb{R})$.

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